Note that, for time-invariant systems, Eq. (8) is exact except for the truncation error, which is of order $O(\Delta t^{K+1})$.

We now find the recursive relation for $P(t)$. Let

$$G_i = \sum_{k=0}^{K_i} D^k(t_{i+\frac{1}{2}}) \frac{\Delta t^k}{k!} \tag{13}$$

From Eq. (8), we have

$$\begin{bmatrix} x(t_{i+1}) \\ \psi(t_{i+1}) \end{bmatrix} = \begin{bmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{bmatrix} \begin{bmatrix} x(t_i) \\ \psi(t_i) \end{bmatrix} \tag{14}$$

It follows from Eqs. (14) and (5) that

$$P(t_{i+1}) = [G_i^{21} + G_i^{22}P(t_i)][G_i^{11} + G_i^{12}P(t_i)]^{-1} \tag{15}$$

or

$$P(t_i) = -[G_i^{22} - P(t_{i+1})G_i^{12}]^{-1}[G_i^{21} - P(t_{i+1})G_i^{11}] \tag{16}$$

Equations (15) and (16) are suitable for initial and terminal conditions (2) and (3), respectively. Equation (15) is to be used with Kalman filtering and Eq. (16) with quadratic regulation. Note that, because of the structure of Eqs. (15) and (16), $P(t)$ will be computed on all of $[t_0, t_f]$ irrespective of the presence of singularities in the solution. This is because $P(t)$ may assume large computed values near the singularities but will not exceed the floating point limit of the computer.

In order to control the size of G_i given by Eq. (13), we could normalize G_i such that its maximal element is one without affecting relations (15) and (16).

Examples

All the computations for the examples were performed on a VAX 11/730 in double precision with $\Delta t = 0.01$.

Example 1. With

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \sqrt{2}/5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & e^{-t} - 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{4} - \frac{1}{2}\sin 2t \end{bmatrix} \tag{17}$$

and

$$P(0) = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \tag{18}$$

the solution is computed using (15) and shown in Fig. 1. The result is identical to Fig. 12 of Ref. 2.

Example 2. Consider an example from Ref. 2 with a singular solution. In this case,

$$A = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2}\cos 2t \\ -\frac{1}{2}\sin 2t & -1 & 0 \end{bmatrix}$$

$$B = -\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 + \frac{1}{2}\sin 2t \end{bmatrix}$$

$$C = \begin{bmatrix} -e^{-t/2} & 0 & 0 \\ 0 & -e^{-t/2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{19}$$

with initial condition $P(0) = -0.4\ I$. The results are shown in Fig. 2. Note that this figure is very similar to Fig. 13 of Ref. 2, where nonlinear superposition formulas are used to solve the matrix Riccati equations. In this case we have a singularity at $t = 0.66$.

These two examples have time-varying elements. The procedure for time-invariant systems is similar and much more accuracy can be expected in the steady-state solution for this case by making K larger in Eq.(8).

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Roll Motion of a Wraparound Fin Configuration at Subsonic and Transonic Mach Numbers

Young Hoon Kim*
Agency of Defense Development
Republic of Korea
and
G.L. Winchenbach†
Air Force Armament Laboratory
Eglin Air Force Base, Florida

Nomenclature

A	= reference area
C_ℓ	= total roll moment coefficient, $\ell/\frac{1}{2}\rho V^2 A d$
$C_{\ell_{p+}}, C_{\ell_{p-}}$	= roll damping derivatives in the clockwise direction, $\partial C_\ell/\partial(p_+ d/2V)$, and counterclockwise direction, $\partial C_\ell/\partial(p_- d/2V)$, respectively
$C_{\ell_{pV}}$	= variation of roll deceleration due to velocity change
C_{ℓ_0}	= total roll driving moment coefficient
C_{ℓ_δ}	= roll moment coefficient due to fin cant
$C_{\ell_{0V}}, C_{\ell_{0V^2}}$	= variations of roll driving moment due to linear and squared velocity changes, respectively
$C_{\ell_{\gamma\bar{\alpha}^2}}$	= induced roll moment derivative
d	= reference diameter
I_x	= axial moment of inertia
N	= number of fins
p	= missile spin rate
$\bar{q}$	= dynamic pressure, $\rho V^2/2$
V	= total missile velocity
V_{ref}	= reference velocity
$\bar{\alpha}$	= missile total angle of attack, $\sin^{-1}[\sqrt{v^2 + w^2}/V]$
δ	= fin cant angle
ξ	= $\sin\bar{\alpha}$
γ	= aerodynamic roll angle, $\tan^{-1}[v/w]$
ϕ	= roll angle
ρ	= air density

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*Exchange Scientist

†Chief, Aeroballistics Section.

Introduction

During a recent free-flight research program on a wraparound fin (WAF) configuration (see Fig. 1), it was observed that some of the models displayed unusual rolling and damping characteristics.[1] For this particular set of data, it was demonstrated that the dynamic instability was not caused by the unusual roll motion but instead was related to the presence of a side moment due to pitch. Generally, during the analysis of these data, the theoretical roll profiles matched the measured roll profiles reasonably well. Nevertheless, the analysis routine had difficulty in determining the roll damping derivative (C_{ℓ_p}). Thus, it was decided to investigate these rolling motions further and attempt to modify the roll moment expansion such that the aerodynamic coefficients could be more precisely determined.[2]

Approach

A summary of the previous fit results is presented in Table 1. The four Mach number groupings consist of two subsonic groups ($M=0.585$ and 0.773), one transonic group ($M=1.062$), and one supersonic group ($M=1.296$). Note that the tabulated C_{ℓ_p} values appear inconsistent and actually could not be determined at the supersonic condition and that these supersonic flights were near the classic pitch-roll resonance condition ($P/\omega_N \approx 1.0$).

The differential equation governing the roll motion[3] is

$$\ddot{\phi} = (\bar{q}Ad/Ix)\{C_\ell(\bar{\alpha},\delta,P,V,\gamma)\} + \tan\theta(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) + \dot{\theta}\dot{\psi}/\sin\theta \tag{1}$$

The mathematical expansion of the total roll moment coefficient, $C_\ell(\delta, \bar{\alpha}, P, V, \gamma)$, with respect to the five assumed parameters is the crux of this investigation. It should also be noted that the V, γ, α, θ, and ψ profiles were all input to the reduction routine using the results of the complete six-degrees-of-freedom analysis reported in Ref. 1.

Because of the demonstrated inadequacy of the previous roll moment expansion[4] to model the motion of the WAF configuration, the following modified roll moment expansion was developed:

$$C_\ell = C_{\ell_0} + C_{\ell_{0V}}(V - V_{\text{ref}}) + C_{\ell_{0V^2}}(V - V_{\text{ref}})^2 + C_{\ell_{\gamma\bar{\alpha}^2}}\xi^2\sin N\gamma + (Pd/2V)\{[S_+ C_{\ell_{p+}} + S_- C_{\ell_{p-}}] + C_{\ell_{pV}}(V - V_{\text{ref}})\} \tag{2}$$

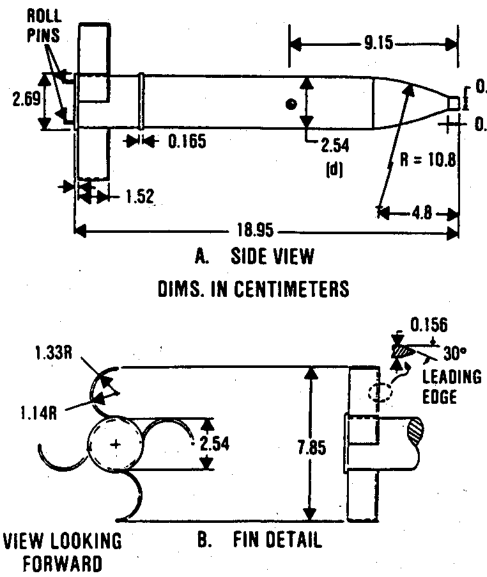

Fig. 1 Wraparound fin configuration.

Five new terms are included in Eq. (2). The C_{ℓ_0} term is the combined driving moment, and $C_{\ell_{0V}}$ and $C_{\ell_{0V^2}}$ terms permit the roll moment to be a strong function of velocity. The other two new terms, $S_+ C_{\ell_{p+}}$ and $S_- C_{\ell_{p-}}$, permit the roll damping derivative to take on different values depending on the direction of roll. This represents a simple spline modeling approach[5] where S_+ and S_- are switches and are set at $S_+ = 1$ and $S_- = 0$ for clockwise motion and $S_+ = 0$ and $S_- = 1$ for counterclockwide motion. The unknown free coefficients in Eq. (2) were determined by fitting the experimentally measured time and angular roll data in association with Eq. (1) using the technique of Chapman and Kirk.[6]

Results and Discussion

The subsonic group used in the present analysis with Eq. (2) as the roll moment expansion contained four flights. Two of these flights were near Mach 0.58 (shots 71 and 72), and two were at about Mach 0.75 (shots 74 and 75). These particular flights were selected because they represented the various types of rolling motion experienced during the test program. Two models rolled clockwise (shots 71 and 72), the flight of shot 74 demonstrated two roll reversals, and the roll motion associated with shot 75 was counterclockwise. The estimated roll moment coefficents obtained from these flights using the modified expansion are shown in Table 2. The various cases shown in this table dramatically demonstrate the effect the various terms of Eq. (2) have on the theoretical fits. Case 1 represents the condition where C_{ℓ_p} is independent of the direction of roll (i.e., $C_{\ell_{p+}} = C_{\ell_{p-}}$). The quality of fit (note listed probable error, PE-ϕ, in Table 2) associated with this case is very poor, indicating that the C_ℓ expansion used is not an adequate representation of the true roll moment. Case 2 is similar to case 1 except that C_{ℓ_p} was allowed to take on two different unique values depending on the direction of roll. The determined values of $C_{\ell_{p+}} = -20.39$ and $C_{\ell_{p-}} = -17.19$ appear reasonable and are consistent with the expected trend; the

Table 1 Results of conventional roll model[1]

Shot numbers	Mach no.	δ^2 $\bar{\alpha}_{\max}$	C_{ℓ_p}	PE-ϕ deg	P/ω_N
71, 72, 73	0.585	8.6 7.6	−22.4	1.19	0.03-0.29
75, 76, 77	0.773	0.3 2.9	−3.47	2.62	0.03-0.45
81, 82	1.062	15.2 7.5	−28.0	30.28	0.05-0.41
87, 88	1.296	40.8 12.1	−5.0[a]	29.71	0.91-1.07

[a]Coefficient could not be determined and was held constant.

Table 2 Effect of various terms (subsonic flights, shots 71, 72, 74, and 75)

	Cases			
	1	2	3	4
$C_{\ell_{p+}}$	−20.17	−20.39	−20.32	−20.09
$C_{\ell_{p-}}$	−20.17	−17.19	−16.70	−17.49
$C_{\ell_{0V}}$	0	0	0.5×10^{-3}	0
$C_{\ell_{0V^2}}$	0	0	0	0
$C_{\ell_{pV}}$	0	0	0	0.003
$C_{\ell_{\gamma\bar{\alpha}^2}}$	0.016	0.013	0.013	0.013
PE-ϕ	39.91 (69.03)	1.56 (1.43)	1.54 (1.36)	1.55 (1.44)

N.B.: ( ) denotes the probable error of shot 75. Counterclockwise rolling motion C_{ℓ_0} term was allowed to be unique for each flight: shot 71 = 0.147; shot 72 = 0.43; shot 74 = 0.007; shot 75 = −0.023.

Table 3 Effect of various terms (transonic flights, shots 81 and 82)

	Cases		
	1	2	3
$C_{\ell_{p+}}$	−30.41	−10.39	−18.29
$C_{\ell_{p-}}$	−30.41	−10.39	−18.29
$C_{\ell_{0V}}$	0	-0.31×10^{-3}	-0.41×10^{-3}
$C_{\ell_{0V^2}}$	0	0	-0.21×10^{-5}
$C_{\ell_{pV}}$	0	0	0
$C_{\ell_{\gamma\bar{\alpha}^2}}$	−0.200	−0.002	−0.810
PE-ϕ	132	4.40	1.27

N.B. C_{ℓ_0} term was allowed to be unique for each flight (see below): shot 81 = −0.045; shot 82 = 0.016.

highest roll damping derivative, $C_{\ell_{p+}} = -20.39$, exists when the fins are cupped into the directon of roll. Also, note that the quality of fit (note PE-ϕ for case 2) is excellent and the listed probable errors are equivalent to the expected measurement precision. The remaining cases (3 and 4) demonstrate the effect of $C_{\ell_{0V}}$ and $C_{\ell_{pV}}$, respectively, on the theoretical fit. Neither of these terms significantly improved or altered the quality of fit.

The two transonic flights (shots 81 and 82) that were reanalyzed during the present investigation were at Mach numbers of 1.028 and 1.092, and both were rolling in the counterclockwise direction. These flights were also fitted using the modified moment expansion, Eq. (2), and the results are shown in Table 3. Case 1 of this table is similar to the original expansion used in Ref. 1, and the quality of fit (PE-ϕ = 132) is very poor, demonstrating that additional terms in the moment expansion are required. Case 2 is similar to case 1 with the addition of the $C_{\ell_{0V}}$ term. This term significantly improved the quality of fit (PE-ϕ = 4.40); however, the probable error is still three times larger than the measuring accuracy associated with these models. The fit represented by case 3 included the quadratic velocity term $C_{\ell_{0V^2}}$ in addition to the other terms, and the associated probable error is consistent with the expected measurement precision (PE-ϕ = 1.27). Also note that C_{ℓ_p} for this case ($C_{\ell_p} = -18.29$) falls between $C_{\ell_{p+}}$ and $C_{\ell_{p-}}$, as determined from the subsonic flights. Since both models were rolling in the counterclockwise direction, the unique values for $C_{\ell_{p+}}$ and $C_{\ell_{p-}}$ could not be determined.

All attempts to reanalyze the measured roll profiles of the two supersonic flights (shots 87 and 88) were unsuccessful. As noted previously, both these flights were near the classic pitch-roll resonant condition ($P/\omega_N \approx 1.0$), and for this condition the angular and rolling motions are strongly coupled. Since the analysis technique used herein assumed that the motions could be decoupled and fitted separately, this may be the cause of the failure to fit the two supersonic flights adequately.

Concluding Remarks

A modified expansion for the rolling moment coefficient has been developed and was used successfully in fitting experimental flight data for a wraparound fin configuration. It is believed that the results obtained at the subsonic and transonic conditions add significantly to the understanding of the rolling motions associated with these configurations. These results indicate that different values of the roll damping derivative C_{ℓ_p} exist, depending on the direction of spin and also that the roll driving moment C_{ℓ_0} is a strong function of velocity throughout the transonic regime.

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• "Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits," Vol. 9, No. 1, 1986, pp. 17-22. In Fig. 4 on p. 19 the number of impulses for the $\beta = 0$ and TIME = 0.4 case should be labelled as 2 rather than 3. The case of $\beta = 270$ does require 3 impulses as indicated.